① (a) $y' = e^{2x} + \sin(x)$

(i) ODE

(ii) order 1

(iii) linear since of the form

$e^{2x} + \sin(x)$

$$\underbrace{a_1(x)}_{1} y' + \underbrace{a_0(x)}_{0 \text{ in this case}} y = b(x)$$

---

① (b) $y'' - 4y = 0$

(i) ODE

(ii) order 2

(iii) linear since of the form

$$\underbrace{a_2(x)}_{1} y'' + \underbrace{a_1(x)}_{0} y' + \underbrace{a_0(x)}_{-4} y = \underbrace{b(x)}_{0}$$

(1)(c) $y''' + 3y'' + 4y' + 12y = x^2 + x - 1$

(i) ODE

(ii) order 3

(iii) linear since of the form

$$\underbrace{a_3(x)}_{3} y''' + \underbrace{a_2(x)}_{3} y'' + \underbrace{a_1(x)}_{4} y' + \underbrace{a_0(x)}_{12} y = \underbrace{b(x)}_{x^2 + x - 1}$$

---

(1)(d) $x^2 y''' - 5y'' + \sin(x) y' - 2y = \cos(x) - 2$

(i) ODE

(ii) order 3

(iii) linear since of the form

$$\underbrace{a_3(x)}_{x^2} y''' + \underbrace{a_2(x)}_{-5} y'' + \underbrace{a_1(x)}_{\sin(x)} y' + \underbrace{a_0(x)}_{-2} y = \underbrace{b(x)}_{\cos(x) - 2}$$

① (e) $\dfrac{d^2y}{dx^2} + yx^3\dfrac{dy}{dx} + x^2y = 0$

(i) ODE

(ii) order 2

(iii) not linear because this coefficient, has a y in it.

$$\dfrac{d^2y}{dx^2} + \boxed{yx^3}\dfrac{dy}{dx} + x^2y = 0$$

---

① (f) $\sin(x^2)\, y' + y = x$

(i) ODE

(ii) order 1

(iii) linear since of the form

$$\underbrace{a_1(x)}_{\sin(x^2)}\, y' + \underbrace{a_0(x)}_{1}\, y = \underbrace{b(x)}_{x}$$

① (g) $(2xy - y^3) + e^x \dfrac{dy}{dx} = 0$

(i) ODE

(ii) order 1

(iii) not linear because of the y^3 term

---

① (h) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial t^2}$

(i) PDE

(ii) order 2 (highest order derivative is 2)

(iii) not applicable since not an ODE

② (a) First note that $f_1(x) = e^{2x}$ and $f_2(x) = e^{-2x}$
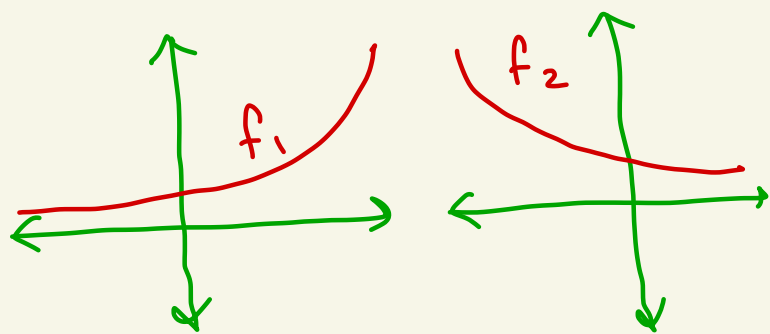are both defined on $I = (-\infty, \infty)$.

We have
$$f_1(x) = e^{2x}$$
$$f_1'(x) = 2e^{2x}$$
$$f_1''(x) = 4e^{2x}$$

all defined on $I = (-\infty, \infty)$

Thus,
$$f_1'' - 4f_1 = 4e^{2x} - 4e^{2x} = 0$$

So, f_1 satisfies $y'' - 4y = 0$

Also,
$$f_2(x) = e^{-2x}$$
$$f_2'(x) = -2e^{2x}$$
$$f_2''(x) = -4e^{2x}$$

all defined on $I = (-\infty, \infty)$

Thus,
$$f_2'' - 4f_2 = -4e^{2x} - 4e^{-2x} = 0$$

So, f_2 satisfies $y'' - 4y = 0$.

② (b)

We know from (a) that f_1 satisfies $y'' - 4y = 0$.

We also have that

$$f_1'(0) = 2e^{2(0)} = 2e^0 = 2$$
$$f_1(0) = e^{2(0)} = e^0 = 1$$

Thus, f_1 satisfies

$$y'' - 4y = 0, \quad y(0) = 1, \quad y'(0) = 2$$

---

② (c)

We know from part (a) that f_2 solves $y'' - 4y = 0$.

We also have that

$$f_2'(1) = -2e^{-2(1)} = -2e^{-2}$$
$$f_2(1) = e^{-2(1)} = e^{-2}$$

Thus, f_2 satisfies

$$y'' - 4y = 0, \quad y(1) = e^{-2}, \quad y'(1) = -2e^{-2}$$

②(d)

Let $f(x) = c_1 f_1(x) + c_2 f_2(x) = c_1 e^{2x} + c_2 e^{-2x}$.

Then,

$$f'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x}$$

$$f''(x) = 4c_1 e^{2x} - 4c_2 e^{-2x}$$

Thus,

$$f'' - 4f = 4c_1 e^{2x} - 4c_2 e^{-2x} + 4\left(c_1 e^{2x} + c_2 e^{-2x}\right) = 0$$

So, f satisfies $y'' - 4y = 0$

---

②(e) We know from part (d) that

$f(x) = c_1 e^{2x} + c_2 e^{-2x}$ satisfies $y'' - 4y = 0$.

We want

$$0 = f'(0) = 2c_1 e^{2(0)} - 2c_2 e^{-2(0)} = 2c_1 - 2c_2$$

$$1 = f(0) = c_1 e^{2(0)} + c_2 e^{2(0)} = c_1 + c_2$$

That is we need to solve

$$\boxed{\begin{array}{l} 2c_1 - 2c_2 = 0 \quad ① \\ c_1 + c_2 = 1 \quad ② \end{array}}$$

which is equivalent to

$$\boxed{\begin{aligned} c_1 - c_2 &= 0 \\ c_1 + c_2 &= 1 \end{aligned}}$$ ①
②

① gives $c_1 = c_2$.

Plug this into ② to get $c_2 + c_2 = 1$.

So, $c_2 = \frac{1}{2}$.

Thus, $c_1 = c_2 = \frac{1}{2}$.

Thus,
$$f(x) = \frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}$$

satisfies
$$y'' - 4y = 0, \quad y'(0) = 0, \quad y(0) = 1$$

③ Let $\varphi(x) = 2\sqrt{x} - \sqrt{x}\ln(x)$

Note that φ is defined for all $x > 0$
that is on $I = (0, \infty)$.

3(a) We have

$\varphi(x) = 2x^{1/2} - x^{1/2}\cdot\ln(x)$

$\varphi'(x) = x^{-1/2} - \frac{1}{2}x^{-1/2}\cdot\ln(x) - x^{1/2}\cdot\frac{1}{x}$

$\quad\quad = x^{-1/2} - \frac{1}{2}x^{-1/2}\cdot\ln(x) - x^{-1/2}$

$\quad\quad = -\frac{1}{2}x^{-1/2}\cdot\ln(x)$

$\varphi''(x) = \frac{1}{4}x^{-3/2}\cdot\ln(x) - \frac{1}{2}x^{-1/2}\cdot\frac{1}{x}$

$\quad\quad = -\frac{1}{2}x^{-3/2} + \frac{1}{4}x^{-3/2}\cdot\ln(x)$

all defined on $I = (0, \infty)$

Thus,

$4x^2\cdot\varphi'' + \varphi = 4x^2\left[-\frac{1}{2}x^{-3/2} + \frac{1}{4}x^{-3/2}\cdot\ln(x)\right]$

$\quad\quad\quad\quad\quad\quad + 2x^{1/2} - x^{1/2}\cdot\ln(x)$

$\quad\quad\quad\quad = -2x^{1/2} + x^{1/2}\cdot\ln(x)$

$\quad\quad\quad\quad\quad\quad + 2x^{1/2} - x^{1/2}\cdot\ln(x) = 0$

Therefore, $\varphi(x) = 2\sqrt{x} - \sqrt{x}\ln(x)$

satisfies $4x^2 y'' + y = 0$ on $I = (0, \infty)$.

3(b) We also have that

$$\varphi(1) = 2\sqrt{1} - \sqrt{1} \cdot \underbrace{\ln(1)}_{0} = 2$$

$$\varphi'(1) = -\frac{1}{2}(1)^{-1/2} \cdot \underbrace{\ln(1)}_{0} = 0$$

Thus, from (a) and the above we know that φ solves the initial-value problem

$$4x^2 y'' + y = 0, \quad y'(1) = 0, \quad y(1) = 2$$

④ Let $f_1(x) = e^{-3x}$, $f_2(x) = \cos(2x)$, $f_3(x) = \sin(2x)$

Note that all three functions are defined
on $I = (-\infty, \infty)$.

Let's start with $f_1(x) = e^{-3x}$

We have
$$f_1(x) = e^{-3x}$$
$$f_1'(x) = -3e^{-3x}$$
$$f_1''(x) = 9e^{-3x}$$
$$f_1'''(x) = -27e^{-3x}$$

all defined
on $I = (-\infty, \infty)$

So,

$$f_1''' + 3f_1'' + 4f_1' + 12f_1$$
$$= -27e^{-3x} + 3[9e^{-3x}] + 4[-3e^{-3x}] + 12[e^{-3x}]$$
$$= -27e^{-3x} + 27e^{-3x} - 12e^{-3x} + 12e^{-3x}$$

$$= 0$$

Thus, $f_1(x) = e^{-3x}$ satisfies

$$y''' + 3y'' + 4y' + 12y = 0$$

Now let's look at $f_2(x) = \cos(2x)$

We have

$f_2(x) = \cos(2x)$

$f_2'(x) = -2\sin(2x)$

$f_2''(x) = -4\cos(2x)$

$f_2'''(x) = 8\sin(2x)$

all defined on $I = (-\infty, \infty)$

And,

$f_2''' + 3f_2'' + 4f_2' + 12f_2$

$= 8\sin(2x) + 3(-4\cos(2x))$
$\quad + 4(-2\sin(2x)) + 12\cos(2x)$

$= 8\sin(2x) - 12\cos(2x)$
$\quad - 8\sin(2x) + 12\cos(2x)$

$= 0$

Thus, $f_2(x) = \cos(2x)$ satisfies

$y''' + 3y'' + 4y' + 12y = 0$

We have

$$f_2(x) = \sin(2x)$$

$$f_2'(x) = 2\cos(2x)$$

$$f_2''(x) = -4\sin(2x)$$

$$f_2'''(x) = -8\cos(2x)$$

all defined on $I = (-\infty, \infty)$

And,

$$f_2''' + 3f_2'' + 4f_2' + 12f_2$$

$$= -8\cos(2x) + 3(-4\sin(2x))$$

$$\qquad + 4(2\cos(2x)) + 12\sin(2x)$$

$$= -8\cos(2x) - 12\sin(2x)$$

$$\qquad + 8\cos(2x) + 12\sin(2x)$$

$$= 0$$

Thus, $f_2(x) = \sin(2x)$ satisfies

$$y''' + 3y'' + 4y' + 12y = 0$$